ON WARPED PRODUCT SUPER GENERALIZED RECURRENT MANIFOLDS

[1]ABSOS ALI SHAIKH, [1]HARADHAN KUNDU AND [2]MD. SHOWKAT ALI

Abstract. The object of the present paper is to obtain the characterization of a warped product semi-Riemannian manifold with a special type of recurrent like structure, called super generalized recurrent. As consequence of this result we also find out the necessary and sufficient conditions for a warped product manifold to satisfy some other recurrent like structures such as weakly generalized recurrent manifold, hyper generalized recurrent manifold etc. Finally as a support of the main result, we present an example of warped product super generalized recurrent manifold.

1. Introduction

To generalize the notion of a manifold of constant curvature Cartan [4] first introduced the notion of local symmetry which can be presented as the curvature restriction $\nabla R = 0$ (i.e., the Riemann-Christoffel curvature tensor R is covariantly constant). But there are many manifolds which does not bear local symmetry and hence to investigate the type of symmetry of such manifolds it is necessary to generalize the notion of local symmetry. During the last eight decades various researchers are working on this area to generalize or extend the notion of local symmetry by weakening its curvature restriction in different directions.

Cartan [5] himself first gave a proper generalization of local symmetry and introduced the notion of semisymmetry, which was later classified by Szabó [35]. Then in 1983 Adamów and Deszcz [1] generalized the notion of semisymmetry and introduced the notion of pseudosymmetry, also known as Deszcz pseudosymmetry (see [20]). On the other hand as a direct generalization of local symmetry, Chaki [6] introduced the notion of pseudosymmetry. We note that the interrelation between two types of pseudosymmetry is studied by Shaikh et. al. [20]. In 1989 Tamássy and Binh [36] generalized the Chaki's notion of pseudosymmetry and introduced the notion of weakly symmetric manifold. We note that Shaikh and his co-authors ([12], [21]-[25]) studied this notion of weak symmetry with various generalized curvature tensors. Again generalizing the results of Binh [2], recently, Shaikh and Kundu [26] obtained the

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characterization of warped product weakly symmetric manifold.

Again recurrent manifold ([16], [17], [18]) is another type of generalization of local symmetry. In 1979 Dubey defined generalized recurrent manifold (briefly, GK_n). Recently [14] Olszak and Olszak showed that every GK_n is concircularly recurrent and every concircularly recurrent manifold is again a K_n and hence every GK_n is a K_n. Again as a generalization of K_n, recently, Shaikh and his coauthors introduced three new type of generalized recurrent structures together with their proper existence, namely, quasi generalized recurrent manifold (briefly, QGK_n) [31], hyper-generalized recurrent manifold (briefly, HGK_n) [30] and weakly generalized recurrent manifold (briefly, WGK_n) [32]. For the existence of such structures we refer the reader to see [19], [34]. Very recently Shaikh et. al. ([27], [33]) introduced another generalization of recurrent manifold, called, super generalized recurrent manifold (briefly, SGK_n) which also generalizes the notion of HGK_n as well as WGK_n. These kinds of generalization of recurrent structures may be called as recurrent like structures.

The main object of the present paper is to obtain the necessary and sufficient condition for a warped product manifold to be HGK_n and WGK_n. For this purpose we first determine the necessary and sufficient condition for a warped product to be SGK_n and as consequence of this result we find out the corresponding results for HGK_n, WGK_n and K_n. We know that decomposable or product manifold is a special case of warped product manifold when the warping function is identically 1. Thus we can present the characterization of a decomposable manifold with various recurrent like structures.

The paper is organized as follows: Section 2 deals with rudimentary facts of various recurrent like structures. Section 3 is concerned with basic curvature relations of a warped product manifold. In section 4 we present our main result and, finally, in section 5 a proper example of a warped product SGK_n is presented.

2. **Preliminaries**

Let M be a non-flat n-dimensional ($n \geq 3$) smooth manifold with semi-Riemannian metric g, Levi-Civita connection be ∇, Riemann-Christoffel curvature tensor R, Ricci tensor S and scalar curvature κ. In this section we define various necessary terms and curvature restricted geometric structures and for this purpose at first we consider some notations:

$C^\infty(M)$ = the algebra of all smooth functions on M,

$\chi(M)$ = the Lie algebra of all smooth vector fields on M,

$\chi^*(M)$ = the Lie algebra of all smooth 1-forms on M and

$\mathcal{T}^r_k(M)$ = the space of all smooth tensor fields of type (r, k) on M.

For $A, E \in \mathcal{T}^0_2(M)$ we have their Kulkarni-Nomizu product [27] $A \wedge E \in \mathcal{T}^0_4(M)$ as

$$A \wedge E_{ijkl} = A_{il}E_{jk} + A_{jk}E_{il} - A_{ik}E_{jl} - A_{jl}E_{ik}. \tag{2.1}$$

In particular we can get $g \wedge g$, $g \wedge S$ and $S \wedge S$ as follows:

$$(g \wedge g)_{ijkl} = 2(g_{il}g_{jk} - g_{ik}g_{jl}),$$

$$(g \wedge S)_{ijkl} = g_{il}S_{jk} + S_{il}g_{jk} - g_{ik}S_{jl} - S_{ik}g_{jl} \text{ and}$$

$$(S \wedge S)_{ijkl} = 2(S_{il}S_{jk} - S_{ik}S_{jl}).$$

Definition 2.1. *The manifold M is said to be recurrent* [37] *if*

$$\nabla R = \Pi \otimes R \quad (\otimes \textit{ denotes the tensor product}) \tag{2.2}$$

$$(\textit{or locally, } R_{ijkl,m} = \Pi_m R_{ijkl}, \quad \textit{where `,' denotes the covariant derivative})$$

holds on $\{x \in M : \nabla R \neq 0$ at $x\}$ for an 1-form $\Pi \in \chi^*(M)$, called the associated 1-form of the recurrent structure. Such an n-dimensional manifold is denoted by K_n.

Again M is said to be concircularly recurrent recurrent if its concircular curvature tensor $W = R - \frac{\kappa}{2n(n-1)} g \wedge g$ satisfies the condition

$$\nabla W = \Pi \otimes W \tag{2.3}$$

on $\{x \in M : \nabla W \neq 0 \text{ at } x\}$ for an 1-form $\Pi \in \chi^*(M)$, called the associated 1-form.
The manifold (M, g) is said to be generalized recurrent [9] if it satisfies

$$\nabla R = \Pi \otimes R + \Theta \otimes g \wedge g \tag{2.4}$$

$$(\text{or locally, } R_{ijkl,m} = \Pi_m R_{ijkl} + 2\Theta_m(g_{il}g_{jk} - g_{ik}g_{jl}))$$

on $\{x \in M : \nabla R \neq \xi \otimes R \text{ at } x \ \forall\ \xi \in \chi^*(M)\}$ for some 1-forms Π and Θ. The 1-forms Π and Θ are called the associated 1-forms of this structure. Such an n-dimensional manifold is denoted by GK_n. In [14] Olszak and Olszak showed that every GK_n satisfying (2.4) is concircularly recurrent with (2.3) and every concircularly recurrent manifold is again a K_n with same associated 1-form and thus $\Theta = 0$. Hence the structure GK_n reduces to K_n. Consequently the notion of GK_n does not exist.

Definition 2.2. *The manifold M is said to be quasi generalized recurrent* [31], *hyper generalized recurrent* [30], *weakly generalized recurrent* [32] *and super generalized recurrent respectively if*

the condition

$$\nabla R = \Pi \otimes R + \Psi \otimes [g \wedge (g + \eta \otimes \eta)]\,, \tag{2.5}$$

$$\nabla R = \Pi \otimes R + \Psi \otimes g \wedge S, \tag{2.6}$$

$$\nabla R = \Pi \otimes R + \Phi \otimes S \wedge S \quad \textit{and} \tag{2.7}$$

$$\nabla R = \Pi \otimes R + \Phi \otimes S \wedge S + \Psi \otimes g \wedge S + \Theta \otimes g \wedge g \tag{2.8}$$

holds respectively on $\{x \in M : \nabla R \neq \xi \otimes R \text{ at } x \;\forall\; \xi \in \chi^*(M)\} \subset M$ *for some* Π, Φ, Ψ, Θ *and* $\eta \in \chi^*(M)$, *called the associated 1-forms.*

An n-dimensional manifold satisfying (2.5) is denoted by QGK_n with (Π, Θ, η) or simply QGK_n. An n-dimensional manifold satisfying (2.6) is denoted by HGK_n with (Π, Ψ) or simply HGK_n. An n-dimensional manifold satisfying (2.7) is denoted by WGK_n with (Π, Φ) or simply WGK_n. An n-dimensional manifold satisfying (2.8) is denoted by SGK_n with $(\Pi, \Phi, \Psi, \Theta)$ or simply SGK_n.

In terms of local coordinates (2.5)-(2.8) can be respectively written as:

$$R_{ijkl,m} = \Pi_m R_{ijkl} + \Theta_m[2(g_{il}g_{jk} - g_{ik}g_{jl}) + g_{il}\eta_j\eta_k + g_{jk}\eta_i\eta_l - g_{ik}\eta_j\eta_l - g_{jl}\eta_i\eta_k], \tag{2.9}$$

$$R_{ijkl,m} = \Pi_m R_{ijkl} + \Psi_m[g_{il}S_{jk} + S_{il}g_{jk} - g_{ik}S_{jl} - S_{ik}g_{jl}], \tag{2.10}$$

$$R_{ijkl,m} = \Pi_m R_{ijkl} + 2\Psi_m[S_{il}S_{jk} - S_{ik}S_{jl}] \quad \text{and} \tag{2.11}$$

$$\begin{aligned} R_{ijkl,m} &= \Pi_m R_{ijkl} + 2\Phi_m[S_{il}S_{jk} - S_{ik}S_{jl}] \\ &+ \Psi_m[g_{il}S_{jk} + S_{il}g_{jk} - g_{ik}S_{jl} - S_{ik}g_{jl}] + 2\Theta[g_{il}g_{jk} - g_{ik}g_{jl}]. \end{aligned} \tag{2.12}$$

It is obvious that the above structures QGK_n, WGK_n and HGK_n are all generalization of K_n but all three exists independently (see [19], [34]). We also mention that SGK_n is a proper generalization of WGK_n and HGK_n (see Section 5).

Definition 2.3. *Let* $D \in \mathcal{T}^0_4(M)$ *and* $A, E, F \in \mathcal{T}^0_2(M)$. *Then* M *is said to be Roter type manifold (briefly,* RT_n*) with* $(D; A, E)$ *(*[7], [8]*) and generalized Roter type manifold (briefly,* GRT_n*) with* $(D; A, E, F)$ *(*[20], [28], [29]*) respectively if*

$$D = N_1 A \wedge A - N_2 A \wedge E - N_3 E \wedge E \quad \textit{and}$$

$$D = L_1 A \wedge A - L_2 A \wedge E - L_3 E \wedge E - L_4 A \wedge F - L_5 E \wedge F - L_6 F \wedge F$$

respectively, for some $N_i, L_j \in C^\infty(M)$, $1 \le i \le 3$ *and* $1 \le j \le 6$.

3. **Warped Product Manifold**

In 1957 Kručkovič [13] initiated the study of semi-decomposable manifolds which were latter named as warped product manifolds by Bishop and O'Neill [3]. Let $(\overline{M}, \overline{g})$ and $(\widetilde{M}, \widetilde{g})$ be two semi-Riemannian manifolds of dimension p and $(n-p)$ respectively $(1 \le p < n)$, and f is a positive smooth function on $\overline{M}$. Then the warped product $M = \overline{M} \times_f \widetilde{M}$ is the product manifold $\overline{M} \times \widetilde{M}$ of dimension n endowed with the metric

$$g = \pi^*(\overline{g}) + (f \circ \pi)\sigma^*(\widetilde{g}), \tag{3.1}$$

where $\pi : M \to \overline{M}$ and $\sigma : M \to \widetilde{M}$ are the natural projections. Then the local components of the metric g are given by

$$g_{ij} = \begin{cases} \overline{g}_{ij} & \text{for } i = a \text{ and } j = b, \\ f\widetilde{g}_{ij} & \text{for } i = \alpha \text{ and } j = \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Here $a, b \in \{1, 2, ..., p\}$ and $\alpha, \beta \in \{p+1, p+2, ..., n\}$. We note that throughout the paper we consider $a, b, c, ... \in \{1, 2, ..., p\}$ and $\alpha, \beta, \gamma, ... \in \{p+1, p+2, ..., n\}$ and $i, j, k, ... \in \{1, 2, ..., n\}$. Here $\overline{M}$ is called the base, $\widetilde{M}$ is called the fiber and f is called warping function of the warped product $M = \overline{M} \times_f \widetilde{M}$. It may be mentioned that the warped product metric g can be taken as (see [11], [13], [38])

$$g = \pi^*(\overline{g}) + (f \circ \pi)^2 \sigma^*(\widetilde{g}).$$

However throughout the paper we will consider the warped product metric given in (3.1). Again we assume that, when Ω is a quantity formed with respect to g, we denote by $\overline{\Omega}$ and $\widetilde{\Omega}$, the similar quantities formed with respect to $\overline{g}$ and $\widetilde{g}$ respectively. By straightforward calculation one can easily calculate the local components of Ω in terms of $\overline{\Omega}$ and $\widetilde{\Omega}$ for $\Omega = \nabla, R, S$ and κ and obtain the following:

The non-zero local components of Levi-Civita connection ∇ of M are given by

$$\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}, \quad \Gamma^a_{\beta\gamma} = -\frac{1}{2}\overline{g}^{ab} f_b \widetilde{g}_{\beta\gamma}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2f} f_a \delta^\alpha_\beta, \tag{3.3}$$

where $f_a = \partial_a f = \frac{\partial f}{\partial x^a}$.

The local components of the Riemann-Christoffel curvature tensor R and Ricci tensor S of M

which may not vanish identically are the following:

$$R_{abcd} = \overline{R}_{abcd}, \quad R_{a\alpha b\beta} = fT_{ab}\widetilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = f\widetilde{R}_{\alpha\beta\gamma\delta} - f^2 P\widetilde{G}_{\alpha\beta\gamma\delta}, \tag{3.4}$$

$$S_{ab} = \overline{S}_{ab} - (n-p)T_{ab}, \quad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta}, \tag{3.5}$$

where $G_{ijkl} = \frac{1}{2} g \wedge g_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$ are the components of Gaussian curvature and

$$T_{ab} = -\frac{1}{2f}(\nabla_b f_a - \frac{1}{2f} f_a f_b), \qquad tr(T) = g^{ab}T_{ab},$$

$$Q = f((n-p-1)P - tr(T)), \qquad P = \frac{1}{4f^2} g^{ab} f_a f_b.$$

The scalar curvature κ of M is given by

$$\kappa = \bar{\kappa} + \frac{\tilde{\kappa}}{f} - (n-p)[(n-p-1)P - 2\, tr(T)]. \tag{3.6}$$

Again the non-zero local components of ∇R are given by [10]:

$$\begin{cases} (i) R_{abcd,e} = \overline{R}_{abcd,e}, \\ (ii) R_{a\alpha b\beta,e} = fT_{ab,e}\widetilde{g}_{\alpha\beta}, \\ (iii) R_{\alpha\beta\gamma\delta,e} = -f_e\widetilde{R}_{\alpha\beta\gamma\delta} + f^2 P_e \widetilde{G}_{\alpha\beta\gamma\delta}, \\ (iv) R_{\alpha\beta\gamma\delta,\epsilon} = f\widetilde{R}_{\alpha\beta\gamma\delta,\epsilon}, \\ (v) R_{\alpha\beta\gamma d,\epsilon} = -\frac{f_d}{2}\widetilde{R}_{\alpha\beta\gamma\epsilon} + \frac{f^2}{2} P_d \widetilde{G}_{\alpha\beta\gamma\epsilon}, \\ (vi) R_{abc\delta,\epsilon} = \frac{1}{2}\widetilde{g}_{\epsilon\delta}(f_a T_{bc} - f_b T_{ac}) + \frac{1}{2} f^d R_{abcd}\widetilde{g}_{\epsilon\delta}. \end{cases} \tag{3.7}$$

The non-zero components of $(g \wedge g)$, $(g \wedge S)$ and $(S \wedge S)$ are given by

$$\begin{cases} (i)(g \wedge g)_{abcd} = (\overline{g} \wedge \overline{g})_{abcd}, \\ (ii)(g \wedge g)_{a\alpha b\beta} = -2f\overline{g}_{ab}\widetilde{g}_{\alpha\beta}, \\ (iii)(g \wedge g)_{\alpha\beta\gamma\delta} = f^2(\widetilde{g} \wedge \widetilde{g})_{\alpha\beta\gamma\delta}, \end{cases} \tag{3.8}$$

$$\begin{cases} (i)(g \wedge S)_{abcd} = (\overline{g} \wedge \overline{S})_{abcd} - (n-p)(\overline{g} \wedge T)_{abcd}, \\ (ii)(g \wedge S)_{a\alpha b\beta} = -\overline{g}_{ab}(\widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta}) - f\widetilde{g}_{\alpha\beta}[\overline{S}_{ab} - (n-p)T_{ab}], \\ (iii)(g \wedge S)_{\alpha\beta\gamma\delta} = f(\widetilde{g} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + fQ(\widetilde{g \wedge g})_{\alpha\beta\gamma\delta}, \end{cases} \tag{3.9}$$

$$\begin{cases} (i)(S \wedge S)_{abcd} = (\overline{S} \wedge \overline{S})_{abcd} - (n-p)(\overline{S} \wedge T)_{abcd} + (n-p)^2(T \wedge T)_{abcd}, \\ (ii)(S \wedge S)_{a\alpha b\beta} = -2(\widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta})[\overline{S}_{ab} - (n-p)T_{ab}], \\ (iii)(S \wedge S)_{\alpha\beta\gamma\delta} = (\widetilde{S} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + Q(\widetilde{g \wedge S})_{\alpha\beta\gamma\delta} + Q^2(\widetilde{g} \wedge \widetilde{g})_{\alpha\beta\gamma\delta}. \end{cases} \tag{3.10}$$

For detailed information about the local components of various tensors on a warped product manifold we refer the reader to see [15], [26], [29] and also references therein.

4. Warped Product SGK_n

Theorem 4.1. *Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product manifold. Then M is a SGK_n with $(\Pi, \Phi, \Psi, \Theta)$ if and only if the following conditions hold simultaneously*

$$(4.1)\quad \begin{cases} (i)\overline{\nabla R} = \overline{\Pi} \otimes \overline{R} + \overline{\Phi} \otimes \overline{S} \wedge \overline{S} + \overline{\Psi} \otimes \overline{g} \wedge \overline{S} + \overline{\Theta} \otimes \overline{g} \wedge \overline{g} \\ \qquad -2(n-p)\overline{\Phi} \otimes \overline{S} \wedge T + (n-p)^2\overline{\Phi} \otimes T \wedge T - (n-p)\overline{\Psi} \otimes \overline{g} \wedge T, \\ (ii) - \widetilde{\Pi} \otimes \overline{R} = \widetilde{\Phi} \otimes \overline{S} \wedge \overline{S} + \widetilde{\Psi} \otimes \overline{g} \wedge \overline{S} + \widetilde{\Theta} \otimes \overline{g} \wedge \overline{g} \\ \qquad -2(n-p)\widetilde{\Phi} \otimes \overline{S} \wedge T + (n-p)^2\widetilde{\Phi} \otimes T \wedge T - (n-p)\widetilde{\Psi} \otimes \overline{g} \wedge T, \end{cases}$$

$$(4.2)\quad \begin{cases} (i) - (df + f\overline{\Pi}) \otimes \widetilde{R} = \overline{\Phi} \otimes \widetilde{S} \wedge \widetilde{S} + (2Q\overline{\Phi} + f\overline{\Psi}) \otimes \widetilde{g} \wedge \widetilde{S} \\ \qquad +(-\frac{1}{2}f^2(P\overline{\Pi} + dP) + Q^2\overline{\Phi} + fQ\overline{\Psi} + f^2\overline{\Theta}) \otimes \widetilde{g} \wedge \widetilde{g}, \\ (ii) f\widetilde{\nabla}\widetilde{R} = f\widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Phi} \otimes \widetilde{S} \wedge \widetilde{S} + (Q\widetilde{\Phi} + f\widetilde{\Psi}) \otimes \widetilde{g} \wedge \widetilde{S} \\ \qquad +(-\frac{1}{2}f^2P\widetilde{\Pi} + Q^2\widetilde{\Phi} + fQ\widetilde{\Psi} + f^2\widetilde{\Theta}) \otimes \widetilde{g} \wedge \widetilde{g}, \end{cases}$$

$$(4.3)\quad \begin{cases} (i)\ \left[2\overline{\Phi} \otimes (\overline{S} - (n-p)T) + \overline{\Psi} \otimes \overline{g}\right] \otimes \widetilde{S} = \\ \quad -[f(\overline{\nabla}T - \overline{\Pi} \otimes T) + (2Q\overline{\Phi} + f\overline{\Psi}) \otimes (\overline{S} - (n-p)T) + (Q\overline{\Psi} + 2f\overline{\Theta}) \otimes \overline{g}] \otimes \widetilde{g}, \\ (ii)\ \left[2\widetilde{\Phi} \otimes (\overline{S} - (n-p)T) + \widetilde{\Psi} \otimes \overline{g}\right] \otimes \widetilde{S} = \\ \quad -[-f\widetilde{\Pi} \otimes T + (2Q\widetilde{\Phi} + f\widetilde{\Psi}) \otimes (\overline{S} - (n-p)T) + (Q\widetilde{\Psi} + 2f\widetilde{\Theta}) \otimes \overline{g}] \otimes \widetilde{g}, \end{cases}$$

$$(4.4)\quad \begin{cases} (i) f^d R_{abcd} = -(f_a T_{bc} - f_b T_{ac}) \quad and \\ (ii) df \otimes \widetilde{R} = f^2 \Theta P \otimes \widetilde{G}. \end{cases}$$

Proof: First suppose that M is SGK_n. Then in terms of local coordinates the defining condition can be written as

$$(4.5)\qquad R_{ijkl,m} = \Pi_m R_{ijkl} + \Phi_m (S \wedge S)_{ijkl} + \Psi_m (g \wedge S)_{ijkl} + \Theta_m (g \wedge g)_{ijkl}.$$

Putting

$$\begin{cases} (i) i = a, j = b, k = c, l = d, m = e \quad \text{and} \\ (ii) i = a, j = b, k = c, l = d, m = \epsilon \end{cases}$$

respectively in (4.5) and then in view of (3.4)-(3.10) it is easy to check that (4.1) holds. Similarly putting

$$\begin{cases} (iii) i=\alpha, j=\beta, k=\gamma, l=\delta, m=e; \\ (iv) i=\alpha, j=\beta, k=\gamma, l=\delta, m=\epsilon; \\ (v) i=a, j=\alpha, k=b, l=\beta, m=e; \\ (vi) i=a, j=\alpha, k=b, l=\beta, m=\epsilon; \\ (vii) i=a, j=b, k=c, l=\alpha, m=\epsilon \text{ and} \\ (viii) i=\alpha, j=\beta, k=\gamma, l=a, m=\epsilon \end{cases}$$

respectively in (4.5) and then in view of (3.4)-(3.10) we get (4.2)-(4.4) respectively. The converse part is obvious. This proves the theorem.

From Theorem 4.1, it follows that the nature of base and fiber of a warped product SGK_n is given by the following:

Corollary 4.1. *Let* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *be a warped product* SGK_n*with* $(\Pi, \Phi, \Psi, \Theta)$. *Then*
(i) $\overline{M}$ *is a* SGK_p *if* T *can be expressed as a linear combination of* $\overline{S}$ *and* $\overline{g}$.
(ii) $\overline{M}$ *is a* GRT_p *with* $(\overline{R}; \overline{g}, \overline{S}, T)$ *on* $\{x \in M : \widetilde{\Pi}_x \neq 0\}$.
(iii) $\widetilde{M}$ *is a* SGK_{n-p} *with* $(\widetilde{\Pi}, \frac{1}{f}\widetilde{\Phi}, \frac{Q}{f}\widetilde{\Phi} + \widetilde{\Psi}, -\frac{1}{2}fP\widetilde{\Pi} + \frac{Q^2}{f}\widetilde{\Phi} + Q\widetilde{\Psi} + f\widetilde{\Theta})$.
(iv) $\widetilde{M}$ *is a* RT_{n-p} *with* $(\widetilde{R}; \widetilde{g}, \widetilde{S})$ *on* $\{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.
(v) $\widetilde{M}$ *satisfies Einstein metric condition on* $\{x \in M : (2(\overline{\kappa} - (n-p)tr(T))\overline{\Phi} + p\overline{\Psi})_x \neq 0\} \cup \{x \in M : (2(\overline{\kappa} - (n-p)tr(T))\widetilde{\Phi} + p\widetilde{\Psi})_x \neq 0\} = \{x \in M : (2(\overline{\kappa} - (n-p)tr(T))\Phi + p\Psi)_x \neq 0\}$.
(vi) $\widetilde{M}$ *is of constant curvature on* $\{x \in M : df_x \neq 0\}$.

From Theorem 4.1, the characterization of a decomposable or product SGK_n is given by the following:

Corollary 4.2. *Let* $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ *be a product manifold. Then* M *is a* SGK_n *with* $(\Pi, \Phi, \Psi, \Theta)$ *if and only if*
1.(i) $\overline{\nabla R} = \overline{\Pi} \otimes \overline{R} + \overline{\Phi} \otimes \overline{S} \wedge \overline{S} + \overline{\Psi} \otimes \overline{g} \wedge \overline{S} + \overline{\Theta} \otimes \overline{g} \wedge \overline{g}$,
(ii) $-\widetilde{\Pi} \otimes \overline{R} = \widetilde{\Phi} \otimes \overline{S} \wedge \overline{S} + \widetilde{\Psi} \otimes \overline{g} \wedge \overline{S} + \widetilde{\Theta} \otimes \overline{g} \wedge \overline{g}$,
2.(i) $\widetilde{\nabla} \widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Phi} \otimes \widetilde{S} \wedge \widetilde{S} + \widetilde{\Psi} \otimes \widetilde{g} \wedge \widetilde{S} + \widetilde{\Theta} \otimes \widetilde{g} \wedge \widetilde{g}$,
(ii) $-\overline{\Pi} \otimes \widetilde{R} = \overline{\Phi} \otimes \widetilde{S} \wedge \widetilde{S} + \overline{\Psi} \otimes \widetilde{g} \wedge \widetilde{S} + \overline{\Theta} \otimes \widetilde{g} \wedge \widetilde{g}$,
3.(i) $\left[2\overline{\Phi} \otimes \overline{S} + \overline{\Psi} \otimes \overline{g}\right] \otimes \widetilde{S} = -[\overline{\Psi} \otimes \overline{S} + 2\overline{\Theta} \otimes \overline{g}] \otimes \widetilde{g}$,
(ii) $\left[2\widetilde{\Phi} \otimes \widetilde{S} + \widetilde{\Psi} \otimes \widetilde{g}\right] \otimes \overline{S} = -[\widetilde{\Psi} \otimes \widetilde{S} + 2\widetilde{\Theta} \otimes \widetilde{g}] \otimes \overline{g}$.

From the above, the nature of each factor of a product SGK_n is given by the following:

Corollary 4.3. *Let* $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ *be a product* SGK_n *with* $(\Pi, \Phi, \Psi, \Theta)$. *Then*
(i) base and fiber are both super generalized recurrent manifolds.

(ii) $\overline{M}$ *is a* RT_p *with* $(\overline{R}; \overline{g}, \overline{S})$ *on* $\{x \in M : \widetilde{\Pi}_x \neq 0\}$.

(iii) $\widetilde{M}$ *is a* RT_{n-p} *with* $(\widetilde{R}; \widetilde{g}, \widetilde{S})$ *on* $\{x \in M : \overline{\Pi}_x \neq 0\}$.

(iv) $\overline{M}$ *satisfies Einstein metric condition on* $\{x \in M : (2\widetilde{\kappa}\widetilde{\Phi} + (n-p)\widetilde{\Psi})_x \neq 0\}$.

(v) $\widetilde{M}$ *satisfies Einstein metric condition on* $\{x \in M : (2\overline{\kappa}\overline{\Phi} + p\overline{\Psi})_x \neq 0\}$.

Corollary 4.4. [26] *Let* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *be a warped product manifold. Then* M *is a recurrent manifold with*

$$\nabla R = \Pi \otimes R$$

if and only if the following conditions hold simultaneously

1.*(i)* $\overline{\nabla R} = \overline{\Pi} \otimes \overline{R}$, *(ii)* $\widetilde{\Pi} \otimes \overline{R} = 0$,

2.*(i)* $-(df + f\overline{\Pi}) \otimes \widetilde{R} = \frac{1}{2} f^2 (P\overline{\Pi} - dP) \otimes \widetilde{g} \wedge \widetilde{g}$, *(ii)* $\widetilde{\nabla}\widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R}$ *and* $P\widetilde{\Pi} = 0$,

3.*(i)* $\overline{\nabla} T = \overline{\Pi} \otimes T$, *(ii)* $\widetilde{\Pi} \otimes T = 0$,

4.*(i)* $f^d R_{abcd} = -(f_a T_{bc} - f_b T_{ac})$ *and (ii)* $df \otimes \widetilde{R} = f^2 dP \otimes \widetilde{G}$.

Corollary 4.5. *Let* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *be a warped product recurrent manifold satisfying* $\nabla R = \Pi \otimes R$. *Then*

(i) $\overline{M}$ *and* $\widetilde{M}$ *are both recurrent. Also* T *is recurrent with associated 1-form* Π.

(ii) $\overline{R} = 0$, $T = 0$ *and* $P = 0$ *on the set* $\{x \in M : \widetilde{\Pi} \neq 0\}$.

(iii) $\widetilde{M}$ *is of constant curvature on* $\{x \in M : df_x \neq 0\} \cup \{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.

Corollary 4.6. *Let* $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ *be a product manifold. Then* M *is recurrent satisfying*

$$\nabla R = \Pi \otimes R$$

if and only if

(i) $\overline{\nabla R} = \overline{\Pi} \otimes \overline{R}$, $\widetilde{\Pi} \otimes \overline{R} = 0$, *(ii)* $\overline{\Pi} \otimes \widetilde{R} = 0$, $\widetilde{\nabla}\widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R}$.

Corollary 4.7. *Let* $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ *be a warped product manifold. Then* M *is a* HGK_n *with* (Π, Ψ) *if and only if the following conditions hold simultaneously*

1.*(i)* $\overline{\nabla R} = \overline{\Pi} \otimes \overline{R} + \overline{\Psi} \otimes \overline{g} \wedge \overline{S} - (n-p)\overline{\Psi} \otimes \overline{g} \wedge T$,

(ii) $-\widetilde{\Pi} \otimes \overline{R} = \widetilde{\Psi} \otimes \overline{g} \wedge \overline{S} - (n-p)\widetilde{\Psi} \otimes \overline{g} \wedge T$,

2.*(i)* $-(df + f\overline{\Pi}) \otimes \widetilde{R} = f\overline{\Psi} \otimes \widetilde{g} \wedge \widetilde{S} + (-\frac{1}{2} f^2 (P\overline{\Pi} + dP) + fQ\overline{\Psi}) \otimes \widetilde{g} \wedge \widetilde{g}$,

(ii) $f\widetilde{\nabla}\widetilde{R} = f\widetilde{\Pi} \otimes \widetilde{R} + f\widetilde{\Psi} \otimes \widetilde{g} \wedge \widetilde{S} + (-\frac{1}{2} f^2 P\widetilde{\Pi} + fQ\widetilde{\Psi}) \otimes \widetilde{g} \wedge \widetilde{g}$,

3.*(i)* $\overline{\Psi} \otimes \overline{g} \otimes \widetilde{S} = -[f(\overline{\nabla} T - \overline{\Pi} \otimes T) + f\overline{\Psi} \otimes (\overline{S} - (n-p)T) + Q\overline{\Psi} \otimes \overline{g}] \otimes \widetilde{g}$,

(ii) $\widetilde{\Psi} \otimes \overline{g} \otimes \widetilde{S} = -[-f\widetilde{\Pi} \otimes T + f\widetilde{\Psi} \otimes (\overline{S} - (n-p)T) + Q\widetilde{\Psi} \otimes \overline{g}] \otimes \widetilde{g}$,

4.*(i)* $f^d R_{abcd} = -(f_a T_{bc} - f_b T_{ac})$ *and*

(ii) $df \otimes \widetilde{R} = f^2 dP \otimes \widetilde{G}$.

Corollary 4.8. *Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product HGK_n with (Π, Ψ). Then*
(i) $\overline{M}$ is hyper generalized recurrent manifold if T and S are linearly dependent.
(ii) $\widetilde{M}$ is a SGK_{n-p} with $(\widetilde{\Pi}, 0, \widetilde{\Psi}, -\frac{1}{2}fP\widetilde{\Pi} + Q\widetilde{\Psi})$.
(iii) $\widetilde{M}$ is of vanishing conformal curvature tensor on $\{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.
(iv) $\widetilde{M}$ satisfies Einstein metric condition on $\{x \in M : \Psi_x \neq 0\}$ and $\widetilde{M}$ is of constant curvature on $\{x \in M : df_x \neq 0\}$.

Proof: Results of (i), (ii) and (iv) are obvious from Corollary 4.7. By virtue of 2.(i) of Corollary 4.7, on $\{x \in M : (df + f\overline{\Pi})_x \neq 0\}$, $\widetilde{R}$ can be exressed as a linear combination of $\widetilde{g} \wedge \widetilde{S}$ and $\widetilde{g} \wedge \widetilde{g}$. Hence in view of Corollary 6.1 of [27], the conformal curvature tensor of $\widetilde{M}$ vanishes on this set, which proves (iii).

Corollary 4.9. *Let $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ be a product manifold. Then M is a HGK_n with (Π, Ψ) if and only if the following conditions hold simultaneously*

1.(i) $\overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R} + \overline{\Psi} \otimes \overline{g} \wedge \overline{S}$, *(ii)* $\widetilde{\Pi} \otimes \overline{R} + \widetilde{\Psi} \otimes \overline{g} \wedge \overline{S} = 0$,
2.(i) $\overline{\Pi} \otimes \widetilde{R} + \overline{\Psi} \otimes \widetilde{g} \wedge \widetilde{S} = 0$, *(ii)* $\widetilde{\nabla}\widetilde{R} = \widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Psi} \otimes \widetilde{g} \wedge \widetilde{S}$,
3.(i) $\overline{\Psi} \otimes \overline{g} \otimes \widetilde{S} = -\overline{\Psi} \otimes \overline{S} \otimes \widetilde{g}$, *(ii)* $\widetilde{\Psi} \otimes \overline{g} \otimes \widetilde{S} = -\widetilde{\Psi} \otimes \overline{S} \otimes \widetilde{g}$.

Corollary 4.10. *Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product manifold. Then M is a WGK_n with (Π, Φ) if and only if the following conditions hold simultaneously*

1.(i) $\overline{\nabla}\overline{R} = \overline{\Pi} \otimes \overline{R} + \overline{\Phi} \otimes \overline{S} \wedge \overline{S} - 2(n-p)\overline{\Phi} \otimes (\overline{S} \wedge T) + (n-p)^2\overline{\Phi} \otimes T \wedge T$,
(ii) $-\widetilde{\Pi} \otimes \overline{R} = \widetilde{\Phi} \otimes \overline{S} \wedge \overline{S} - 2(n-p)\widetilde{\Phi} \otimes (\overline{S} \wedge T) + (n-p)^2\widetilde{\Phi} \otimes T \wedge T$,
2.(i) $-(df + f\overline{\Pi}) \otimes \widetilde{R} = \overline{\Phi} \otimes \widetilde{S} \wedge \widetilde{S} + 2Q\overline{\Phi} \otimes \widetilde{g} \wedge \widetilde{S} + (-\frac{1}{2}f^2(P\overline{\Pi} - dP) + Q^2\overline{\Phi}) \otimes \widetilde{g} \wedge \widetilde{g}$,
(ii) $f\widetilde{\nabla}\widetilde{R} = f\widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Phi} \otimes \widetilde{S} \wedge \widetilde{S} + Q\widetilde{\Phi} \otimes \widetilde{g} \wedge \widetilde{S} + (-\frac{1}{2}f^2P\widetilde{\Pi} + Q^2\widetilde{\Phi}) \otimes \widetilde{g} \wedge \widetilde{g}$,
3.(i) $2\overline{\Phi} \otimes (\overline{S} - (n-p)T) \otimes \widetilde{S} = -[f(\overline{\nabla}T - \overline{\Pi} \otimes T) + 2Q\overline{\Phi} \otimes (\overline{S} - (n-p)T)] \otimes \widetilde{g}$,
(ii) $2\widetilde{\Phi} \otimes (\overline{S} - (n-p)T) \otimes \widetilde{S} = -[-f\widetilde{\Pi} \otimes T + 2Q\widetilde{\Phi} \otimes (\widetilde{S} - (n-p)T)] \otimes \widetilde{g}$,
4.(i) $f^dR_{abcd} = -(f_aT_{bc} - f_bT_{ac})$ *and*
(ii) $df \otimes \widetilde{R} = f^2dP \otimes \widetilde{G}$.

Corollary 4.11. *Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product WGK_n with (Π, Φ). Then*
(i) $\overline{M}$ is a WGK_p if T and S are linearly dependent.
(ii) $\widetilde{M}$ is a SGK_{n-p} with $(\widetilde{\Pi}, \Phi, \frac{Q}{f}\widetilde{\Psi}, -\frac{1}{2}fP\widetilde{\Pi} + \frac{Q^2}{f}\widetilde{\Phi})$.
(iii) $\widetilde{M}$ is a RT_{n-p} with $(\widetilde{R}; \widetilde{g}, \widetilde{S})$ on $\{x \in M : (df + f\overline{\Pi})_x \neq 0\}$.
(iv) $\widetilde{M}$ satisfies Einstein metric condition on $\{x \in M : ((\overline{\kappa} - (n-p)tr(T))\Phi)_x \neq 0\}$ and is of constant curvature on $\{x \in M : df_x \neq 0\}$.

Corollary 4.12. *Let $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ be a product manifold. Then M is a WGK_n with (Π, Φ) if and only if the following conditions hold simultaneously*

1.(i) $\overline{\nabla R} = \overline{\Pi} \otimes \overline{R} + \overline{\Phi} \otimes \overline{S} \wedge \overline{S}$, (ii) $\widetilde{\Pi} \otimes \overline{R} + \widetilde{\Phi} \otimes \overline{S} \wedge \overline{S} = 0$,
2.(i) $\overline{\Pi} \otimes \widetilde{R} + \overline{\Phi} \otimes \widetilde{S} \wedge \widetilde{S} = 0$, (ii) $\widetilde{\nabla R} = \widetilde{\Pi} \otimes \widetilde{R} + \widetilde{\Phi} \otimes \widetilde{S} \wedge \widetilde{S}$,
3.(i) $\overline{\Phi} \otimes \overline{S} \otimes \widetilde{S} = 0$, (ii) $\widetilde{\Phi} \otimes \overline{S} \otimes \widetilde{S} = 0$.

5. **An example of warped product** SGK_4

Example 1: Consider the warped product $M = \overline{M} \times_f \widetilde{M}$, where $\overline{M}$ is a 3-dimensional manifold equipped with the metric

$$\overline{ds}^2 = e^{x^2}(dx^1)^2 + e^{x^1}(dx^2)^2 + (dx^3)^2$$

in local coordinates (x^1, x^2, x^3) and $\widetilde{M}$ is an open interval of $\mathbb{R}$ with local coordinate x^4 and the warping function $f = e^{x^3}$. The non-zero components of the Riemann-Christoffel curvature tensor $\overline{R}$ and Ricci tensor $\overline{S}$ of $\overline{M}$ are given by

$$\overline{R}_{1212} = -\frac{1}{4}\left(e^{x^1} + e^{x^2}\right), \ \overline{S}_{11} = \frac{1}{4}\left(1 + e^{x^2 - x^2}\right), \ \overline{S}_{22} = \frac{1}{4}\left(e^{x^1 - x^2}\right).$$

Again the non-zero components of $\overline{\nabla R}$ are

$$\overline{R}_{1212,1} = \frac{e^{x^2}}{4}, \ \overline{R}_{1212,2} = \frac{e^{x^1}}{4}.$$

If we consider the 1-form $\overline{\Pi} = \left(-\frac{e^{x^2}}{e^{x^1}+e^{x^2}}, -\frac{e^{x^1}}{e^{x^1}+e^{x^2}}, 0\right)$, then we can easily check that the manifold $\overline{M}$ is recurrent satisfying $\overline{\nabla R} = \overline{\Pi} \otimes \overline{R}$.
Now the metric of $M = \overline{M} \times_f \widetilde{M}$ is given by

$$ds^2 = e^{x^2}(dx^1)^2 + e^{x^1}(dx^2)^2 + (dx^3)^2 + e^{x^3}(dx^4)^2.$$

The non-zero local components of the Riemann-Christoffel curvature tensor R and Ricci tensor S of M are given by

$$R_{1212} = -\frac{1}{4}\left(e^{x^1} + e^{x^2}\right), \ R_{3434} = -\frac{e^{x^3}}{4},$$
$$S_{11} = \frac{1}{4}\left(e^{x^2 - x^1} + 1\right), \ S_{22} = \frac{1}{4}\left(e^{x^1 - x^2} + 1\right), \ S_{33} = \frac{1}{4}, \ S_{44} = \frac{e^{x^3}}{4}.$$

Again the non-zero local components of ∇R are given by

$$R_{1212,1} = \frac{e^{x^2}}{4}, \ R_{1212,2} = \frac{e^{x^1}}{4}.$$

Again the non-zero components of $g \wedge g$, $g \wedge S$ and $S \wedge S$ given as follows:

$$g \wedge g_{1212} = -2e^{x^1 + x^2}, \ g \wedge g_{1313} = -2e^{x^2}, \ g \wedge g_{1414} = -2e^{x^2 + x^3}, \ g \wedge g_{2323} = -2e^{x^1},$$

$$g \wedge g_{2424} = -2e^{x^1 + x^3}, \ g \wedge g_{3434} = -2e^{x^3},$$

$$g\wedge S_{1212}=\frac{1}{2}\left(-e^{x^1}-e^{x^2}\right),\ g\wedge S_{1313}=\frac{1}{4}\left(-e^{x^2-x^1}\left(e^{x^1}+1\right)-1\right),$$
$$g\wedge S_{1414}=-\frac{1}{4}e^{x^3-x^1}\left(e^{x^2}\left(e^{x^1}+1\right)+e^{x^1}\right),\ g\wedge S_{2323}=\frac{1}{4}\left(-e^{x^1-x^2}\left(e^{x^2}+1\right)-1\right),$$
$$g\wedge S_{2424}=-\frac{1}{4}e^{x^3-x^2}\left(e^{x^2}\left(e^{x^1}+1\right)+e^{x^1}\right),\ g\wedge S_{3434}=-\frac{e^{x^3}}{2},$$
$$S\wedge S_{1212}=\frac{1}{4}\left(-\cosh\left(x^1-x^2\right)-1\right),\ S\wedge S_{1313}=\frac{1}{8}\left(-e^{x^2-x^1}-1\right),$$
$$S\wedge S_{1414}=-\frac{1}{8}e^{x^3}\left(e^{x^2-x^1}+1\right),\ S\wedge S_{2323}=\frac{1}{8}\left(-e^{x^1-x^2}-1\right),$$
$$S\wedge S_{2424}=-\frac{1}{8}e^{x^3}\left(e^{x^1-x^2}+1\right),\ S\wedge S_{3434}=-\frac{e^{x^3}}{8}.$$

If we consider the 1-forms Π, Φ, Ψ and Θ as:

$$\Pi_i=\begin{cases}\frac{\Psi_1e^{x^1}\left(e^{x^2}-2\right)+2\Psi_1\cosh(x^1-x^2)-(2\Psi_1+1)e^{x^2}+2\Psi_1}{2\left(e^{x^1}+e^{x^2}\right)} & \text{for } i=1\\ \frac{\Psi_2\left(e^{x^1}-2\right)e^{x^2}+2\Psi_2\cosh(x^1-x^2)-(2\Psi_2+1)e^{x^1}+2\Psi_2}{2\left(e^{x^1}+e^{x^2}\right)} & \text{for } i=2\\ \frac{\Psi_3e^{-x^1-x^2}\left(-e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)^2}{2\left(e^{x^1}+e^{x^2}\right)} & \text{for } i=3\\ \frac{\Psi_4e^{-x^1-x^2}\left(-e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)^2}{2\left(e^{x^1}+e^{x^2}\right)} & \text{for } i=4,\end{cases}\tag{5.1}$$

$$\Phi_i=\begin{cases}\frac{\left(\Psi_1\left(-e^{x^1+x^2}\right)+2\Psi_1\cosh(x^1-x^2)+2\Psi_1+e^{x^2}\right)}{-2\cosh(x^1-x^2)+\sinh(x^1)+\cosh(x^1)+\sinh(x^2)+\cosh(x^2)-2} & \text{for } i=1\\ e^{x^1}\left(\frac{\Psi_2e^{x^1}+1}{e^{x^1}+e^{x^2}}-\Psi_2+\frac{1}{\left(e^{x^1}-1\right)e^{x^2}-e^{x^1}}\right)-\Psi_2 & \text{for } i=2\\ -\frac{\Psi_3\left(e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)}{e^{x^1}+e^{x^2}} & \text{for } i=3\\ -\frac{\Psi_4\left(e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)}{e^{x^1}+e^{x^2}} & \text{for } i=4,\end{cases}\tag{5.2}$$

$$\Theta_i=\begin{cases}\frac{-\Psi_1e^{x^1-x^2}+2\Psi_1e^{x^2}\sinh(x^1)-2\Psi_1+e^{x^2}}{16\left(-e^{x^1+x^2}+e^{x^1}+e^{x^2}\right)} & \text{for } i=1\\ \frac{1}{16}\left(-\Psi_2e^{-x^1}-\Psi_2e^{-x^2}-\Psi_2+\frac{e^{x^1}}{-e^{x^1+x^2}+e^{x^1}+e^{x^2}}\right) & \text{for } i=2\\ -\frac{1}{16}\Psi_3e^{-x^1-x^2}\left(e^{x^1+x^2}+e^{x^1}+e^{x^2}\right) & \text{for } i=3\\ -\frac{1}{16}\Psi_4e^{-x^1-x^2}\left(e^{x^1+x^2}+e^{x^1}+e^{x^2}\right) & \text{for } i=4\end{cases}\tag{5.3}$$

then we can check that M is a SGK_4 with (Π,Φ,Ψ,Θ), which is neither a HGK_4 nor a WGK_4.

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[1]Department of Mathematics,
University of Burdwan, Golapbag,
Burdwan-713104,
West Bengal, India

E-mail address: aask2003@yahoo.co.in

E-mail address: kundu.haradhan@gmail.com

[2]Department of Mathematics,
Dhaka University,
Dhaka-1000, Bangladesh.

E-mail address: msa317@yahoo.com